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EXACT SOLUTIONS OF THE ONE-DIMENSIONAL RUSSO–SMEREKA KINETIC EQUATION

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We obtain new classes of invariant solutions of the integrodifferential equations describing the propagation of nonlinear concentration waves in a rarefied bubbly fluid. For all the solutions obtained, trajectories of particle motion in phase space are calculated. The stability of some flows is studied in a linear approximation. In several cases, the construction of solutions reduces to an integrodifferential equation of the second kind, which can be solved by the iteration method.

Kinetic approaches based on the statistical description of the interaction of a large number of bubbles are often used for modeling of concentration waves in a flow of a bubbly fluid. A recent result in this area is the kinetic model of a rarefied bubbly flow derived by Russo and Smereka.

The present work deals with the construction of exact particular solutions of the one-dimensional Russo–Smereka equation by methods of the classical group analysis. Simpler submodels determining the families of exact solutions are obtained using a group of admissible point transformations, and some of these submodels are integrated. Physical interpretation of the solutions obtained is given.

1. Mathematical Model and Admissible Transformations. Kinetic equations of motion of bubbles in a fluid were derived and exploited in [1–3] and in a number of other works. In [4], Russo and Smereka proposed an integrodifferential model that describes the propagation of concentration waves in a rarefied bubbly fluid. In this model, the bubbles are rigid massless spheres of the same radius, the fluid is inviscid, incompressible, and at rest at infinity, and its flow in the region between the bubbles is irrotational. In dimensionless variables, the one-dimensional Russo–Smereka equation is given by [5]

$$f_t + (p - j)f_x + pj_x f_p = 0, \qquad j(t, x) = \int_{-\infty}^{\infty} pf\, dp. \tag{1.1}$$

Here t is time, x is the spatial variable, p is the momentum of a bubble, $f(t, x, p)$ is the unknown distribution function for the bubbles in phase space, and $j(t, x)$ is the first moment of the distribution function.

The model is adequate for the description of real flows of a rarefied bubbly fluid in the case of small pressure variations. The condition for a bubbly flow to be rarefied is given by the inequality

$$n(t, x) = \int_{-\infty}^{\infty} f(t, x, p)\, dp < 1.$$

We seek solutions of Eq. (1.1) for which this condition is satisfied. Teshukov [5] studied the characteristic properties of (1.1) and traveling waves and constructed an infinite series of conservation laws.

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We note that Eq. (1.1) is invariant with respect to the following group of transformations G_4: 1) $t' = t + a$; 2) $x' = x + a$; 3) $t' = at$ and $x' = ax$; 4) $x' = ax$, $p' = ap$, and $f' = a^{-1}f$. These transformations correspond to the Lie algebra of operators L_4: $X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = t\partial_t + x\partial_x$, and $X_4 = x\partial_x + p\partial_p - f\partial_f$. The method developed in [6] allows one to construct invariant solutions of Eq. (1.1) using subalgebras of L_4.

For efficient use of these transformations for finding the invariant solutions, the optimal system of subalgebras of the Lie algebra of operators L_4 is derived using the algorithm proposed in [7]. The list of all representatives of the optimal system of rank 1 is follows: 1) $\alpha X_3 + X_4$; 2) $X_1 + X_4$; 3) $X_2 - X_3 + X_4$; 4) X_3; 5) $X_1 + X_2$; 6) X_2; 7) X_1. The system is optimal in the sense that the classes of solutions obtained using its representatives give, up to change of variables, all possible invariant solutions that correspond to one-parameter subgroups of the transformation group G_4. Subsequent construction of invariant solutions reduces to determination of invariants of the corresponding subalgebras and the derivation and integration of quotient systems.

2. Submodels. For all representatives of the optimal system of rank one, we give sets of basis invariants J, representations of solutions, and quotient systems E/H [$H(\alpha^i X_i)$ refers to a subalgebra].

1. $H(\alpha X_3 + X_4)$: $J = (t^{-(1+\beta)}x, t^{-\beta}p, t^{\beta}f)$, and $\beta = \alpha^{-1}$. The solution is invariant with respect to dilation of all variables that depend on the parameter α ($\alpha \neq -1, 0$). This solution describes a class of self-similar (in the restricted sense) motions of the medium. The solution is written as

$$\xi = t^{-(1+\beta)}x, \quad \varphi = t^{-\beta}p, \quad f = t^{-\beta}\psi(\xi,\varphi), \quad j = t^{\beta}m(\xi).$$

The quotient system E/H is given by

$$-\beta\psi + (\varphi - m - (1+\beta)\xi)\psi_\xi + (m_\xi - \beta)\varphi\psi_\varphi = 0, \quad m(\xi) = \int\limits_{-\infty}^{\infty} \varphi\psi\, d\varphi. \tag{2.1}$$

For $\alpha = 0$, we have $J = (t, x^{-1}p, xf)$. The solution is invariant with respect to dilation of x, p and f. The solution is written as

$$\varphi = x^{-1}p, \qquad f = x^{-1}\psi(t,\varphi), \qquad j = xm(t).$$

The quotient system E/H is given by

$$\psi_t - (\varphi - m)\psi + (2m - \varphi)\varphi\psi_\varphi = 0, \qquad m(t) = \int\limits_{-\infty}^{\infty} \varphi\psi\, d\varphi.$$

2. $H(X_1 + X_4)$: $J = (x\exp(-t), p\exp(-t), f\exp(t))$. The solution is invariant with respect to simultaneous translation in t and dilation of x, p, and f. The solution is written as

$$\xi = x\exp(-t), \quad \varphi = p\exp(-t), \quad f = \exp(-t)\psi(\xi,\varphi), \quad j = \exp(t)m(\xi).$$

The quotient system E/H is given by

$$-\psi + (\varphi - m - \xi)\psi_\xi + (m_\xi - 1)\varphi\psi_\varphi = 0, \qquad m(\xi) = \int\limits_{-\infty}^{\infty} \varphi\psi\, d\varphi.$$

3. $H(X_2 - X_3 + X_4)$: $J = (t\exp(x), tp, t^{-1}f)$. The solution is invariant with respect to simultaneous translation in the x direction and dilation of t, p, and f. The solution is written as

$$\xi = t\exp(x), \qquad \varphi = tp, \qquad f = t\psi(\xi,\varphi), \qquad j = t^{-1}m(\xi).$$

The quotient system E/H is given by

$$\psi + (\varphi - m + 1)\xi\psi_\xi + (1 + \xi m_\xi)\varphi\psi_\varphi = 0, \quad m(\xi) = \int\limits_{-\infty}^{\infty} \varphi\psi\, d\varphi. \tag{2.2}$$

4. $H(X_3)$; $J = (t^{-1}x, p, f)$. The solution is invariant with respect to uniform dilation of variables t and x and describes a class of self-similar motions of the medium. The solution is written as $\xi = t^{-1}x$ and $f = f(\xi, p)$. The quotient system E/H is given by

$$(p - j - \xi)f_\xi + pj_\xi f_p = 0, \qquad j(\xi) = \int_{-\infty}^{\infty} pf\,dp. \tag{2.3}$$

5. $H(X_1 + X_2)$; $J = (x - t, p, f)$. The solution is invariant with respect to simultaneous translation in t and x and describes traveling waves. The solution is written as $\xi = x - t$ and $f = f(\xi, p)$. The quotient system E/H is given by

$$(p - j - 1)f_\xi + pj_\xi f_p = 0, \qquad j(\xi) = \int_{-\infty}^{\infty} pf\,dp. \tag{2.4}$$

6. $H(X_2)$; $J = (t, p, f)$. The solution is invariant with respect to translation along the x axis. The solution is written as $f = f(t, p)$ and $j = j(t)$. The quotient system E/H is given by

$$f_t = 0. \tag{2.5}$$

7. $H(X_1)$; $J = (x, \lambda, p, f)$. The solution is invariant with respect to translation in time. The solution is written as $f = f(x, p)$. The quotient system E/H is given by

$$(p - j)f_x + pj_x f_p = 0, \qquad j(x) = \int_{-\infty}^{\infty} pf\,dp. \tag{2.6}$$

3. Invariant Solutions. Below we present results of integration of the quotient systems and analyze the solutions obtained.

The trajectories of bubble motion in the phase space are obtained from the system of ordinary differential equations

$$\frac{dx}{dt} = p - j, \qquad \frac{dp}{dt} = pj_x. \tag{3.1}$$

The linear stability of the flows is studied using the characteristic equation

$$\chi(k) = 1 - n + (j + k)^2 \int_{-\infty}^{\infty} \frac{f_p}{p - j - k}\,dp = 0 \tag{3.2}$$

and the hyperbolicity conditions

$$\Delta \arg \chi^\pm(p) = 0, \qquad \chi^\pm(p) \neq 0,$$

$$\chi^\pm(p) = 1 - n(t, x) + p^2\Big(\int_{-\infty}^{\infty} \frac{\partial f(t, x, p')}{\partial p'}\,\frac{1}{p' - p}\,dp' \pm \pi i\,\frac{\partial f(t, x, p)}{\partial p}\Big)$$

(the increment of the argument is calculated with variation in p from $-\infty$ to ∞ at fixed t and x), which are obtained in [5] by the method proposed in [8]. The hyperbolicity conditions guarantee that Eq. (3.2) has no complex characteristic roots, and these conditions are necessary for flow stability.

We note that for the particular class of solutions with $n = 1$, Eq. (1.1) reduces to equations that describe plane-parallel rotational flows of an inviscid homogeneous fluid in a long channel. The characteristic properties of this system are analyzed in [9], where some exact solutions are also given.

Submodel (2.5). Integration of the quotient system (2.5) yields a class of steady, spatially homogeneous solutions $f = f(p)$.

Submodels (2.4) *and* (2.6). Integration of the quotient systems (2.4) and (2.6) yields the inequalities

$$f = \Phi(p^2 - 2(j + a)p), \qquad j(x - at) = \int_{-\infty}^{\infty} p\Phi\, dp$$

($a = 1$ and 0, respectively). Up to a change of notation, these values of parameter a completely describe the class of invariant solutions of the form $f = f(x - Dt, p)$, where $D = \text{const}$ (traveling waves). These solutions are considered in [5], where formulas describing a traveling wave propagating with constant velocity D over a spatially homogeneous background are given.

Free Motion of Bubbles in an Inviscid Incompressible Fluid. *Submodel* (2.2). We consider the case of $m = 1$. The characteristic system of equations (2.2) has the first integrals $\varphi - \ln \xi$ and $\varphi\psi$. Therefore, a solution of the quotient system can be written as

$$\psi = \varphi^{-1} A(\varphi - \ln \xi), \qquad \int_{-\infty}^{\infty} A(\varphi - \ln \xi)\, d\varphi = 1$$

(A is an arbitrary function). We construct a solution in a domain $-\infty < \varphi < \infty$, $\exp(b) < \xi < \infty$ ($b = \text{const}$). To do this, as A we take any nonnegative differentiable function defined in the interval $[-b, \infty)$, which, together with its derivative, vanishes at the point $-b$ and at infinity and satisfies the equality $\int_{-b}^{\infty} A(\lambda)\, d\lambda = 1$. Outside the interval $[-b, \infty)$, we continue the function $A(\lambda)$ by the zero function. As a result, in the domain $b < \ln t + x < \infty$, we obtain the solution of Eq. (1.1)

$$f(t, x, p) = p^{-1} A(tp - x - \ln t), \quad 0 < q < p < \infty, \qquad f(t, x, p) = 0, \quad -\infty < p \leqslant q,$$

$$\int_{q}^{\infty} A(tp - x - \ln t)\, dp = t^{-1} \qquad (q = (-b + \ln t + x)t^{-1}). \tag{3.3}$$

For solutions of the class (3.3), $j = t^{-1}$ and the function $n(t, x)$ attains a maximum value on the curve $b \doteq \ln t + x$ and decreases monotonically with increase in t and x.

In the Russo-Smereka kinetic model, the force exerted on a bubble system is proportional to the gradient of the first moment of the distribution function. In this case, $j_x = 0$ for all times, and, therefore, solutions (3.3) describe free motion of bubbles in an inviscid incompressible fluid. We note that the absence of the forces is related to the special self-consistent distribution of the bubbles in space. From Eqs. (3.1), we find that the trajectories of the bubbles are given by the formulas

$$x = tp_0 - \ln t - x_0, \qquad p = p_0 \tag{3.4}$$

(x_0 and p_0 are constants).

We consider an example of a solution of the class (3.3). Let

$$A(\lambda) = \frac{2\alpha}{\pi} \cos^2(\alpha\lambda), \qquad \lambda \in \Big[-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\Big];$$

otherwise $A(\lambda) = 0$ (α is a positive constant). Then, in the domain $\pi/(2\alpha) < \ln t + x < \infty$, we obtain a solution with the finite distribution function

$$f(t, x, p) = \frac{2\alpha}{\pi p} \cos^2(\alpha(tp - \ln t - x)) \tag{3.5}$$

if $-\pi/(2\alpha) + \ln t + x \leqslant tp \leqslant \pi/(2\alpha) + \ln t + x$; otherwise $f \equiv 0$. Let, for definiteness, $\alpha = 0.14$.

Figure 1 shows bubble distributions in space at fixed times. It follows from Eqs. (3.5) and Fig. 1 that in a bounded range of x, the support of the distribution function is contracted in the variable p with time. At large t, the function $f \neq 0$ only in a small interval $p \in (0, \varepsilon)$ [$\varepsilon = (\pi/(2\alpha) + \ln t + x)t^{-1}$ and goes to zero as $t \to \infty$]. Therefore, in the process of flow evolution, only bubbles whose momenta are nearly zero remain in the observation domain. To explain this fact, we consider the bubble trajectories (3.4). It follows

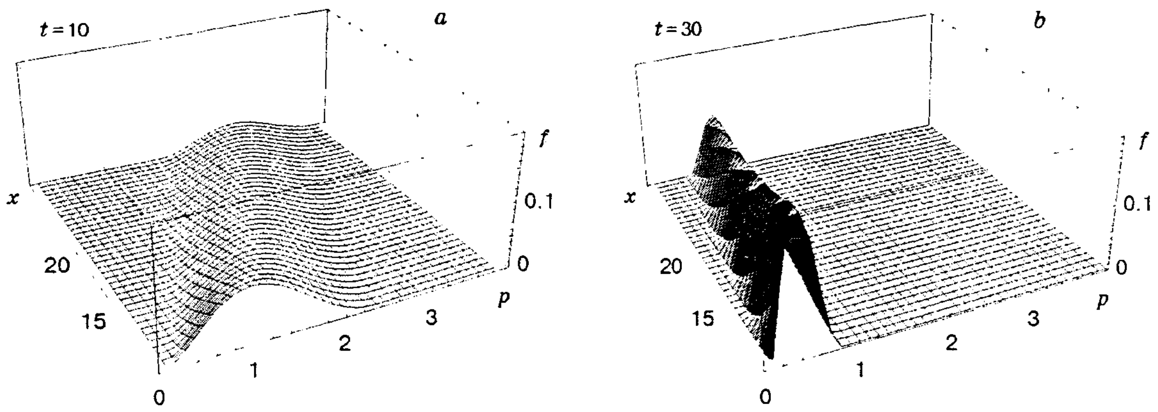

Fig. 1

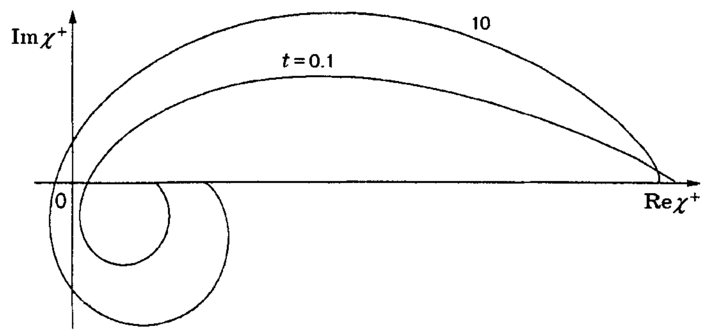

Fig. 2

from Eqs. (3.4) that, starting from a certain time, every bubble moves in the direction of increasing values of x since $p = p_0 > 0$ ($f \equiv 0$ for $p < 0$) and $x \sim p_0 t$, but the bubbles differ in velocity. Bubbles with large momenta move faster (and leave the observation domain faster) than bubbles with small momenta. Thus, this solution describes the process of free "scattering" of bubbles in an inviscid incompressible fluid. We note, that in this case, the analogue of the hydrodynamic density $n(t, x) < 1$ and decreases with time (this is verified by straightforward calculations).

We now study the linear stability of the flow (3.5) using the characteristic equation, i.e., the hyperbolicity conditions, which guarantee that Eq. (3.2) has no complex roots for a given solution. Figure 2 shows plots of the function $\chi^+(p)$ for p varied from $-\infty$ to $+\infty$ at times $t = 0.1$ and 10 at the point $x = \pi\alpha^{-1} \approx 22.439$; the values of $\operatorname{Re}\chi^+$ are plotted on the abscissa axis, and the values of $\operatorname{Im}\chi^+$ are on the ordinate axis. The plots of the functions $\chi^+(p)$ and $\chi^-(p)$ are symmetric about the abscissa axis. It follows from Fig. 2 that, at $t = 0.1$, the increment of the argument of the functions $\chi^\pm$ is zero, and the hyperbolicity conditions are satisfied. Therefore, in a certain neighborhood of the point $x = \pi\alpha^{-1}$, the flow is stable in the linear approximation at $t \approx 0.1$. At $t = 10$ (Fig. 2), $\Delta \arg \chi^+(p) = 2\pi$ and $\Delta \arg \chi^-(p) = -2\pi$. In this case, the hyperbolicity conditions are violated (there are complex characteristic roots) and the flow is unstable. Thus, we have shown that for certain initial data, instability can appear in a free motion of the bubbles.

Flows of a Rarefied Bubbly Fluid with a Critical Layer. Below, we present results concerning self-similar solutions of the Russo–Smereka equation.

Submodel (2.3). We consider particular solutions of the quotient system (2.3) for which $m = \sigma\xi$ (σ is an arbitrary constant). Let $\sigma \neq -1$ and 0 [for $\sigma = -1$, we obtain solutions for which $n = 1$, and for $\sigma = 0$, solutions have the form $f = f(p)$]. Integration of (2.3) yields

$$f = \Phi(C), \quad C = |\xi| \left|\frac{(1-2\gamma)p}{\xi}\right|^{1-\gamma} \left|1 - \frac{(1-2\gamma)p}{\xi}\right|^{\gamma}, \quad \frac{\gamma}{1-2\gamma}\xi = \int p\Phi\, dp, \tag{3.6}$$

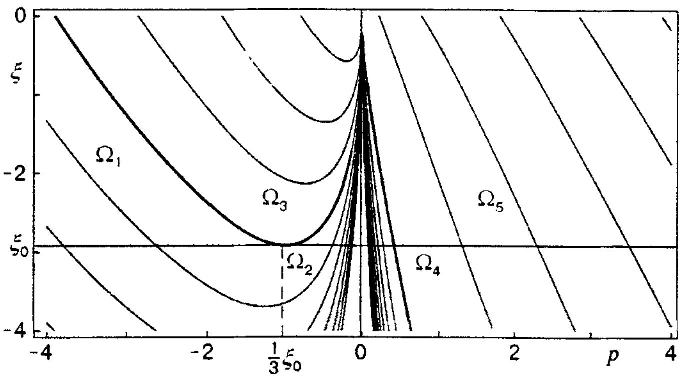

Fig. 3

where $\gamma = \sigma(1+\sigma)^{-1}$. The distribution function is constant along the curves $C = \text{const}$. Let $\gamma = 2$ (the other cases are similar but the mathematical calculations are more complicated). Then, according to (3.6), the invariant C has the form

$$C = |p|^{-1}(3p+\xi)^2. \tag{3.7}$$

Figure 3 shows the characteristics (curves $C = \text{const}$) on the plane (p,ξ). We consider the following Cauchy problem:

$$f(\xi_0,p) = f_0(p), \qquad \gamma(1-2\gamma)^{-1}\xi_0 = \int_{-\infty}^{\infty} pf_0(p)\,dp. \tag{3.8}$$

Let us construct a solution similar to a simple wave in the domain $-\infty < p < \infty$, $\xi_0 < \xi < \xi_1 < 0$. Conditions (3.8) guarantee the continuous matching of the simple wave with the specified steady, spatially homogeneous solution $f_0(p)$. As one can see in Fig. 3, for $\xi > \xi_0$, the solution of the Cauchy problem is uniquely determined from the initial data in the domains Ω_1, Ω_2, Ω_4, and Ω_5. In the domain Ω_3, bounded by the bold-face curve $C = C_0 = -12\xi_0$ and the straight line $\xi = \xi_1$, the solution is determined using additional equations. We note that the Cauchy problem (3.8) is ill-posed for $\xi < \xi_0$ because the function $f_0(p)$ cannot be specified arbitrarily (the characteristics intersect the curve with prescribed Cauchy conditions at two points).

Let us find a solution in the domains

$$\Omega_1 = \{(p,\xi)\colon \xi_0 \le \xi < \xi_1,\ -\infty < p \leqslant (2\xi_0 - \xi - 2\sqrt{\xi_0^2 - \xi_0\xi})/3\},$$

$$\Omega_2 = \{(p,\xi)\colon \xi_0 \leqslant \xi < \xi_1,\ (2\xi_0 - \xi + 2\sqrt{\xi_0^2 - \xi_0\xi})/3 \leqslant p \leqslant 0\}.$$

For that, we calculate the function $\Phi(C)$ $(C \geqslant C_0)$ at $\xi = \xi_0$ (we denote it by Φ_{01} for $p \leqslant \xi_0/3$ and Φ_{02} for $p \geqslant \xi_0/3$). From Eqs. (3.7) and (3.8), we obtain

$$\Phi_{01}(C) = f_0((-6\xi_0 - C - \sqrt{12\xi_0 C + C^2})/18), \quad \Phi_{02}(C) = f_0((-6\xi_0 - C + \sqrt{12\xi_0 C + C^2})/18).$$

Using the known functions Φ_{01} and Φ_{02}, from (3.6) and (3.7) we obtain the following solution in the domains Ω_1 and Ω_2:

$$f(p,\xi) = \Phi_1(-(3p+\xi)^2/p),\ -\infty < p \leqslant (2\xi_0 - \xi - 2\sqrt{\xi_0^2 - \xi_0\xi})/3,$$

$$f(p,\xi) = \Phi_2(-(3p+\xi)^2/p),\ (2\xi_0 - \xi + 2\sqrt{\xi_0^2 - \xi_0\xi})/3 \leqslant p \leqslant 0.$$

Then, we obtain the solution in the domains

$$\Omega_4 = \{(p,\xi)\colon \xi_0 \leqslant \xi < \xi_1,\ 0 \leqslant p \leqslant -\xi/3\}, \quad \Omega_5 = \{(p,\xi)\colon \xi_0 \leqslant \xi < \xi_1,\ -\xi/3 \leqslant p < \infty\}.$$

For $0 \leqslant p \leqslant -\xi_0/3$ and $p \geqslant -\xi_0/3$ on the line $\xi = \xi_0$, function $\Phi(C)$ $(0 \leqslant C < \infty)$ is given by
$\Phi_{04}(C) = f_0((-6\xi_0 + C - \sqrt{-12\xi_0 C + C^2})/18)$ and $\Phi_{05}(C) = f_0((-6\xi_0 + C + \sqrt{-12\xi_0 C + C^2})/18)$, respectively. These formulas allow one to determine the simple wave in the domains Ω_4 and Ω_5:

$$f(p,\xi) = \Phi_4((3p+\xi)^2/p), \quad 0 \leqslant p \leqslant -\xi/3, \quad f(p,\xi) = \Phi_5((3p+\xi)^2/p), \quad -\xi/3 \leqslant p < \infty.$$

We now construct a solution in the domain

$$\Omega_3 = \{(p,\xi)\colon \xi_0 \leqslant \xi < \xi_1,\ (2\xi_0 - \xi - 2\sqrt{\xi_0^2 - \xi_0\xi})/3 \leqslant p \leqslant (2\xi_0 - \xi + 2\sqrt{\xi_0^2 - \xi_0\xi})/3\}.$$

We transform the relation

$$-\frac{2\xi}{3} = \int_{-\infty}^{\infty} pf(\xi,p)\,dp \tag{3.9}$$

to an integral equation for the function $\Phi(C)$ in the interval $(C_1 = -12\xi_1 < C < C_0 = -12\xi_0)$. For that, in each domain where the simple wave is already known, we change in (3.9) the variable of integration p to C. Let $s = -12\xi$. In the domain Ω_1, the function f and the variable p can be expressed in terms of s and C:

$$f = \Phi_1(C), \quad p = (s/2 - C - \sqrt{C}\sqrt{C-s})/18, \quad C_0 < C < \infty.$$

Therefore,

$$\int_{-\infty}^{a_1} pf\,dp = 18^{-2}\int_{C_0}^{\infty}\Big(s - 2C - \sqrt{C}\sqrt{C-s} - \frac{(2C-s)^2}{4\sqrt{C}\sqrt{C-s}}\Big)\Phi_1(C)\,dC,$$

where $a_1 = (2\xi_0 - \xi - 2\sqrt{\xi_0^2 - \xi_0\xi})/3$. Integrals in the other domains are transformed similarly. Finally, for the unknown function Φ in the domain Ω_3 we obtain the following integral equation of the second kind:

$$\int_{s}^{C_0} K(C,s)\Phi(C)\,dC = F(s), \tag{3.10}$$

where

$$K = \sqrt{C}\sqrt{C-s} + \frac{(2C-s)^2}{4\sqrt{C}\sqrt{C-s}},$$

$$F(s) = -9s + \frac{1}{2}\int_{C_0}^{\infty}(s - 2C - K(C,s))\Phi_1(C)\,dC - \frac{1}{2}\int_{C_0}^{\infty}(s - 2C + K(C,s))\Phi_2(C)\,dC$$

$$-\frac{1}{2}\int_{0}^{\infty}(s + 2C - V(C,s))\Phi_4(C)\,dC + \frac{1}{2}\int_{0}^{\infty}(s + 2C + V(C,s))\Phi_5(C)\,dC,$$

$$V = \sqrt{C}\sqrt{C+s} + \frac{(2C+s)^2}{4\sqrt{C}\sqrt{C+s}}.$$

We now show that (3.10) can be transformed to an equation of the second kind. We separate the singularity in the kernel of the integral operator (3.10):

$$K(C,s) = \frac{s\sqrt{s}}{4\sqrt{C-s}} + Q(C,s), \quad Q(C,s) = \sqrt{C}\sqrt{C-s} + \frac{(2C-s)^2 - s\sqrt{sC}}{4\sqrt{C}\sqrt{C-s}}.$$

The kernel $Q(C,s)$ has no singularities in the integration domain. Therefore, Eq. (3.10) can be rewritten as

$$\int_{s}^{C_0} \frac{\Phi(C)}{\sqrt{C-s}}\,dC = G(s), \quad G(s) = \frac{4}{s\sqrt{s}}\Big(F(s) - \int_{s}^{C_0} Q(C,s)\Phi(C)\,dC\Big). \tag{3.11}$$

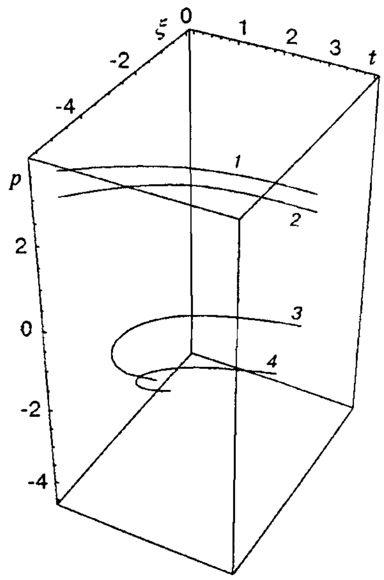

Fig. 4

The function $G(s)$ is continuously differentiable and vanishes at the point $s = C_0$. Inversion of Abel's integral operator (3.11) allows one to obtain the following integral equation of the second kind for the function $\Phi(C)$:

$$\Phi(C) = -\frac{1}{\pi}\int\limits_C^{C_0} \frac{G'(s)}{\sqrt{s-C}}\,ds. \tag{3.12}$$

Equation (3.12) can be solved by the iteration method. If the function $\Phi(C)$ is obtained, the self-similar solution is determined.

Integrating (1.1) over the variable p, we find that the function $n(t,x)$ satisfies the differential equation

$$n_t + ((1-n)j)_x = 0.$$

For the class of simple waves, it has the form $n' + 2(1-n)/\xi = 0$. Solving this equation, we obtain

$$n(\xi) = 1 - (1-n_0)\Big(\frac{\xi}{\xi_0}\Big)^2, \qquad n_0 = \int\limits_{-\infty}^{\infty} f_0(p)\,dp. \tag{3.13}$$

An analysis of (3.13) shows that in a simple wave, the density $n(\xi)$ increases and, in the limit, it reaches unity as $|\xi|$ decreases to zero.

In the variables t, ξ, and p, Eqs. (3.1) are given by

$$\frac{d\xi}{dt} = \frac{3p-\xi}{3t}, \qquad \frac{dp}{dt} = -\frac{2p}{3t} \qquad (t>0) \tag{3.14}$$

and define the trajectories of the bubbles relative to a reference frame moving with the simple wave. Integrating Eq. (3.14), we obtain the trajectories $\xi = -3at^{-2/3} + bt^{-1/3}$ and $p = at^{-2/3}$, where a and b are arbitrary constants. We note that $C = (3p+\xi)^2/|p| = \text{const}$ is an integral of system (3.14). Therefore, we shall use Fig. 3 for analysis of the trajectories. The quantity $p - \xi/3$ is negative in the domain Ω_1, positive in the domains Ω_2, Ω_4, and Ω_5, and changes sign from negative to positive when it crosses the straight line connecting the points $(\xi_0/3, \xi_0)$ and $(0,0)$ on the plane (p,ξ). The solution constructed describes a flow with a critical layer since on the curve $\xi = 3p$, the particle velocity coincides with the wave velocity. Bubbles whose relative velocity changes sign at a certain point of the trajectory penetrate through the front $\xi = \xi_*$ $(\xi_0 \leqslant \xi_* < \xi_1)$ into the simple wave domain Ω_3. After that, these bubbles return to the front $\xi = \xi_*$ and leave

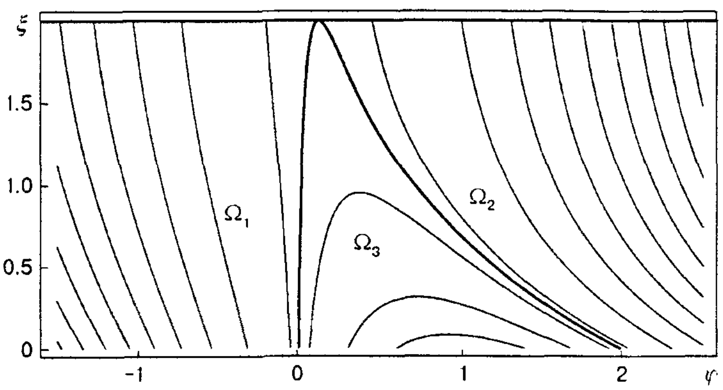

Fig. 5

the domain Ω_3. Bubble trajectories for the simple wave are shown in Fig. 4 in the space of the variables (t, ξ, p). In Fig. 4, the projections of curves 1 and 2 on the plane (p, ξ) are in the domains Ω_4 and Ω_5, and the projections of the curves 3 and 4 (which have turning points) cross the domains Ω_1, Ω_3, and Ω_2. Thus, these self-similar solutions describe the penetration of bubbles in the unperturbed region through which the simple wave propagates.

Submodel (2.1). We consider Eq. (1.2) for $\beta = -1/2$. In this case, the solution and the quotient system E/H are written as

$$\xi = x/\sqrt{t}, \qquad \varphi = p\sqrt{t}, \qquad f = \sqrt{t}\psi(\xi, \varphi); \tag{3.15}$$

$$\frac{\psi}{2} + (\varphi - l)\psi_\xi + l'\varphi\psi_\varphi = 0, \qquad l(\xi) = \int\limits_{-\infty}^{\infty} \varphi\psi\, d\varphi + \frac{\xi}{2}. \tag{3.16}$$

We note that the characteristic system for the first equation in (3.16) has the integral

$$C = \varphi^2 - 2\varphi l(\xi). \tag{3.17}$$

This allows one to find another integral and write ψ in the form

$$\psi = \Phi(C)\exp\left(\mp\frac{1}{2}\int\limits_{\xi_0}^{\xi}\frac{1}{\sqrt{l^2(\tau)+C}}\,d\tau\right). \tag{3.18}$$

In (3.18), we choose the minus if $\varphi - l(\xi) > 0$ and the plus if $\varphi - l(\xi) < 0$. The second equation of (3.16), written as

$$l(\xi) - \frac{\xi}{2} = \int\limits_{-l^2(\xi)}^{\infty}\left(\frac{l(\xi)}{\sqrt{l^2(\xi)+C}}\cosh\left(\frac{1}{2}\int\limits_{\xi_0}^{\xi}\frac{1}{\sqrt{l^2(\tau)+C}}\,d\tau\right)\right.$$

$$\left. - \sinh\left(\frac{1}{2}\int\limits_{\xi_0}^{\xi}\frac{1}{\sqrt{l^2(\tau)+C}}\,d\tau\right)\right)\Phi(C)\,dC. \tag{3.19}$$

is used to determine $\Phi(C)$.

In the interval $[\xi_0, \xi_1]$, we define an arbitrary continuously differentiable, nonnegative, monotonically decreasing function $l(\xi)$. Figure 5 shows the curves $C = \text{const}$ on the plane (φ, ξ). In this case, $l(\xi) = \exp(-\xi)$, $\xi_0 = 0$, and $\xi_1 = 2$. For another choice of $l(\xi)$, Fig. 5 remains qualitatively the same. In the domains

$$\Omega_1 = \{(\xi, \varphi)\colon \xi_0 \leqslant \xi \leqslant \xi_1,\ -\infty < \varphi \leqslant l(\xi) - \sqrt{l^2(\xi) - l_1^2}\},$$

$$\Omega_2 = \{(\xi, \varphi)\colon \xi_0 \leqslant \xi \leqslant \xi_1,\ l(\xi) + \sqrt{l^2(\xi) - l_1^2} \leqslant \varphi < \infty\},$$

the integral C takes values from $-l_1^2$ to infinity, and in the domain

$$\Omega_3 = \{(\xi, \varphi)\colon \xi_0 \leqslant \xi \leqslant \xi_1,\ l(\xi) - \sqrt{l^2(\xi) - l_1^2} < \varphi < l(\xi) + \sqrt{l^2(\xi) - l_1^2}\}$$

it takes values from $-l_0^2$ to $-l_1^2$ [$l_0 = l(\xi_0)$ and $l_1 = l(\xi_1)$].

On the half-line $[-l_1^2, \infty)$, we define the function $\Phi = \Phi_*(C)$ such that Eq. (3.19) is satisfied at the point $\xi = \xi_1$. As one can see in Fig. 5, values of the function $\Phi(C)$ are determined from the known function $\Phi_*(C)$ in the domains Ω_1 and Ω_2 and are not determined in the domain Ω_3. To construct a solution in the domain Ω_3, it is necessary to determine the function $\Phi(C)$ in the interval $[-l_0^2, -l_1^2)$. To do this, we transform Eq. (3.19):

$$\int\limits_{-l^2}^{-l_1^2} \Big(\frac{l}{\sqrt{l^2 + C}} \cosh\left(q(l, C)\right) - \sinh\left(q(l, C)\right)\Big)\Phi(C)\,dC$$

$$= l - \frac{\xi(l)}{2} - \int\limits_{-l_1^2}^{\infty} \Big(\frac{l}{\sqrt{l^2 + C}} \cosh\left(q(l, C)\right) - \sinh\left(q(l, C)\right)\Big)\Phi_*(C)\,dC = F(l), \tag{3.20}$$

where

$$q(l, C) = \frac{1}{2} \int\limits_{l_0}^{l} \frac{\xi'(\zeta)}{\sqrt{\zeta^2 + C}}\,d\zeta$$

is a continuous function of the variables l and C.

We separate the singularity in the kernel of the integral operator (3.20) and introduce the notation $s = -l^2$, $s_1 = -l_1^2$, $s_0 = -l_0^2$, and $\tilde{F}(s) = F(l)$. Thus, we obtain the equation

$$\int\limits_{s}^{s_1} \frac{\Phi(C)}{\sqrt{C - s}}\,dC = G(s) = \frac{1}{\sqrt{-s}\cosh\left(q(\sqrt{-s}, s)\right)}\Big[\tilde{F}(s)$$

$$- \int\limits_{s}^{s_1} \Big(\frac{\sqrt{-s}}{\sqrt{C - s}}\left[\cosh\left(q(\sqrt{-s}, C)\right) - \cosh\left(q(\sqrt{-s}, s)\right)\right] - \sinh\left(q(\sqrt{-s}, C)\right)\Big)\Phi(C)\,dC\Big]. \tag{3.21}$$

The differentiable function $G(s)$ is defined on the interval $[s_0, s_1]$. We invert Abel's integral operator (3.21):

$$\Phi(C) = \frac{1}{\pi}\Big(\frac{G(s_1)}{\sqrt{s_1 - C}} - \int\limits_{C}^{s_1} \frac{G'(s)}{\sqrt{s - C}}\,ds\Big). \tag{3.22}$$

As a result, we have obtained the integral equation of the second kind (3.22) for the function $\Phi(C)$. Equation (3.22) is uniquely solvable by the successive approximation method. We note that $G(s_1) = 0$ because Eq. (3.19) is satisfied at the point $\xi = \xi_1$. If the function $\Phi(C)$ is known, Eqs. (3.15), (3.17), and (3.18) define an invariant solution of Eq. (1.1).

We write Eqs. (3.1), which define the particle trajectories, in variables ξ and φ:

$$\frac{d\xi}{dt} = \frac{\varphi - l(\xi)}{t}, \qquad \frac{d\varphi}{dt} = \frac{l'(\xi)\varphi}{t}.$$

The relative velocity of the bubbles changes sign in the domain Ω_3 when the quantity $\varphi - l(\xi)$ vanishes. Therefore, this invariant solution describes flows with a critical layer.

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